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FORMULATION OF BOUNDARY INTEGRAL EQUATIONS FOR THREE-DIMENSIONAL ELASTO-PLASTIC FLOW†

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Abstract—A general theory of elasto-plastic flow is formulated for work-hardening materials that may be both anisotropic and compressible. Because the theory is quasi-linear, it may be cast in terms of integral equations and the result is an extended form of Somigliana's identity. When these relations are evaluated on the boundary of a solid, their dimensionality is reduced. Previous experience with simpler materials shows that arbitrary problems may be solved in a direct manner.

INTRODUCTION

IN AN increasing number of instances, structural analysts are being asked to provide details of the stress and deformation fields local to geometric irregularities. With the needs for reduced weight, increased integrity and lower costs becoming more demanding, the requisite analyses must be more precise.

These trends have evinced themselves over the last few years and, for the most part, the response has been in the form of numerical methods. Various procedures have been developed by analysts working at all levels, from fairly pure research to quite specifically applied efforts. Of these methods, the finite element technique has evolved to a high level of performance, and its utility in a wide range of situations is broadly accepted.

In the case of bulky, three-dimensional bodies, however, especially when fine resolution is needed, the various finite element procedures are far from optimal. Such problems, when attacked by methods that discretize the entire volume, pose enormous computer requirements in terms of both core size and machine time. Indeed the very sparsity of literature in this area would suggest that such problems are normally treated in an oblique manner.

Recognizing this need, we have given some attention to solution methods especially tailored to such problems. One of us (TAC) has developed both the theory and its numerical implementation to the point where linear elastic problems of both theoretical and technological interest may successfully be studied. The procedure, earlier denoted as the direct potential method, may be viewed as comprising two steps. Once a problem has been set through specification of boundary data, the first computational step is taken. Derived from surface integrals, it produces the full complement of boundary data over the entire surface. The totality of tractions and displacements is then queried for the solution at pre-selected interior points. The second step may be repeated many times to elucidate the information sought at the outset.

While a more detailed description of this approach appears below, one feature is worth noting here. The rather extensive analytical formulation pays off in that the major work of

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problem-solving need be performed only on the boundary of the body under study, not throughout its volume because the dimensionality of the problem is reduced by one, e.g. from three to two.

As this capability has evolved, one of us (JLS) has also built a grounding in the analysis of elasto–plastic flow. This work has involved both theoretical studies and the development of efficient solution methods and, at present, we are able to treat a range of two-dimensional situations on a routine basis. The heart of this approach is in its inclusion of work-hardening materials. Given this feature, the governing equations are elliptic, and any solution method developed originally for the more elementary case of linear elasticity merely needs to be extended in order to treat elasto–plastic flow. We have also been able to show that the solutions generated using this approach are physically realistic, and it is therefore viewed as an accurate model of observed inelastic behavior.

Recently, we have discerned means for merging these two methods and, in this report, we describe the combined formulation. Several elements were developed individually, including an extensive generalization of the theory of elasto–plastic flow, study of its mathematical character, articulation of a reciprocal theorem for quasi-linear behavior and assemblage of the foregoing into an extended form of Somigliana's identity. The individual developments in themselves are of some interest, of course; we believe the important content is the potential for solving a difficult class of problems in a new and direct manner.

Further work is needed to settle fully certain mathematical details, and extensive additional effort is needed to devise and refine computer code implementing the basic idea. Such work is under way, of course, and will be reported separately. The present paper is intended to focus on the basic formulation and certain theoretical developments that were required to achieve it.

ELASTO–PLASTIC FLOW

In previous work [1–5], we have confined our attention to a description of plastic behavior that, for the most part, followed classical lines. Thus, for example, yielding was taken as isotropic and engendered an incompressible strain state. The novel feature of this work is that the equations were cast in a form that allowed examination of their mathematical character [1] and greatly facilitated solution [2, 3]. It happens that far more general equations can be derived at no great increase in complexity, and we follow this pattern here.

In passing, we note the ironic point that extensive experimentation would be required to exploit fully the potential of the theory in the newer form. This matter is of some interest because we have been, and continue to be, concerned with physical behavior of materials. Comparisons between experiment and theory [4, 5] are encouraging but inexact, one apparent reason being the lack of correct material data [6]. Thus, we expect that the theory can and shall be used to design experiments for the proper measurement of material properties, and look forward to the opportunity of so doing.

Governing equations

As in the previous study [1], all processes are presumed to be quasi-static so that both inertial and convective effects may be neglected. Thus for a typical variable

$$\dot{q} = \partial q/\partial t = \mathrm{d}q/\mathrm{d}t.$$

As a result, the strain–displacement relations may be used in the form†

$$\dot{\varepsilon}_{ij} = (\dot{u}_{i,j} + \dot{u}_{j,i})/2 \tag{1}$$

with initial conditions specified. Similarly, the static equilibrium equations

$$\sigma_{ij,i} + X_j = 0\,;\, \sigma_{ij} = \sigma_{ji}$$

become

$$\dot{\sigma}_{ij,j} + \dot{X}_i = 0. \tag{2}$$

If $\dot{X}_i = 0$, then equilibrium is met by

$$\dot{\sigma}_{ij,j} = 0. \tag{2a}$$

The initial conditions for (2) or (2a) require statement. In general, the history of body-force excitation must be specified if X_i causes yielding.

At any instant of time, the instantaneous values of displacements and stresses are taken as known. Attention here is directed towards finding their rates of change as excitation continues. We thus view these rates as the dependent variables in a problem, their integration (with respect to time) being a subsequent operation.

The strain rates are separated into elastic and plastic parts in the usual manner, viz.

$$\dot{\varepsilon}_{ij} = \dot{\varepsilon}_{ij}^{(e)} + \dot{\varepsilon}_{ij}^{(p)}$$

and

$$2\mu\dot{\varepsilon}_{ij}^{(e)} = \dot{\sigma}_{ij} - \nu\dot{\sigma}_{kk}\delta_{ij}/(1+\nu). \tag{3}$$

In (3) μ is the shear modulus, and ν is Poisson's ratio for isotropic elasticity. In the case of anisotropic elasticity, (3) will be of the form

$$\dot{\varepsilon}_{ij}^{(e)} = C_{ijkl}\dot{\sigma}_{kl} \tag{3a}$$

where C_{ijkl} is a compliance tensor having up to 21 independent constants. Without specific values of the compliances, we merely demonstrate the procedure rather than perform the detailed calculations which are carried out for the isotropic case.

To express the plastic strain rates, we take a loading function of the form

$$\phi(\sigma_{ij}) - \psi(\varepsilon_{kl}^{(p)}, W^{(p)}) \leq 0 \tag{4}$$

where the plastic strain energy density is

$$W^{(p)} \equiv W^{(p)}(x,t) = \int_{-\infty}^{t} \sigma_{ij}(x,\tau)\dot{\varepsilon}_{ij}^{(p)}(x,\tau)\,\mathrm{d}\tau.$$

The functions ϕ and ψ are presumed to have the dimensions of stress and to possess first derivatives. Thus ϕ is interpretable as an equivalent stress τ_{eq} (and ψ as that value at which flow may occur). The flow rule associated with (4) is formally

$$\dot{\varepsilon}_{ij}^{(p)} = \frac{(\partial\phi/\partial\sigma_{ij})(\partial\phi/\partial\sigma_{kl})\dot{\sigma}_{kl}}{[(\partial\psi/\partial\varepsilon_{mn}^{(p)}) + (\partial\psi/\partial W^{(p)})\sigma_{mn}](\partial\phi/\partial\sigma_{mn})}.$$

† Standard indicial notation and its associated conventions are used throughout. The range of indices is 1, 2, 3 and δ_{ij} is the Kronecker delta; eq are not indices.

Having defined τ_{eq}, the flow rule may be put in simpler form. We first introduce an equivalent plastic strain rate $\dot{\varepsilon}_{eq}^{(p)}$ such that

$$\tau_{eq}\dot{\varepsilon}_{eq}^{(p)} = \sigma_{ij}\dot{\varepsilon}_{ij}^{(p)} = \dot{W}^{(p)}.$$

The implied functional dependence between τ_{eq} and $\dot{\varepsilon}_{eq}^{(p)}$, together with the quasi-static nature of the process, allows the definition

$$\mathrm{d}\tau_{eq}/\mathrm{d}\varepsilon_{eq}^{(p)} \equiv 2\mu_{eq}^{(p)}$$

where $\mu_{eq}^{(p)}$ is an equivalent plastic modulus, and the expression

$$\dot{\varepsilon}_{eq}^{(p)} = \dot{\tau}_{eq}/2\mu_{eq}^{(p)}$$

follows. Eliminating explicit reference to $\dot{\varepsilon}_{eq}^{(p)}$ in the flow rule gives

$$\dot{\varepsilon}_{ij}^{(p)} = \frac{\phi}{2\mu_{eq}^{(p)}}\frac{(\partial\phi/\partial\sigma_{ij})(\partial\phi/\partial\sigma_{kl})}{\sigma_{mn}\partial\phi/\partial\sigma_{mn}}\dot{\sigma}_{kl}$$

and, if we employ the shorthand notation

$$\phi_{ij} = \partial\phi/\partial\sigma_{ij}$$

the flow rule becomes simply

$$2\mu\dot{\varepsilon}_{ij}^{(p)} = [\mu/\mu_{eq}^{(p)}][\phi\phi_{ij}\phi_{kl}\dot{\sigma}_{kl}/\phi_{mn}\sigma_{mn}]. \tag{5}$$

In appearance, (5) is the same as that derived originally [1].

In substance, however, (5) is somewhat less restrictive. Since ψ is a function of both $\varepsilon_{ij}^{(p)}$ and $W^{(p)}$, the equivalence of their use is obvious in that the flow rule depends only on the parameter $\mu_{eq}^{(p)}$. The function ϕ is limited only by conditions of differentiability and dimensionality; as a result, yielding may be both anisotropic and compressible, should experiment so dictate. It should be noted that two types of anisotropy are possible. The first is that between $\dot{\varepsilon}_{ij}^{(p)}$ and $\dot{\sigma}_{kl}$ for all forms of ϕ, including its dependence solely on stress invariants. The second type is that noted above, wherein ϕ may have directional sensitivity, as will occur in, say, cold-worked metals.

Inasmuch as the rate terms in (5) appear explicitly, the flow rule may be assembled with (3)—or (3a)—to give

$$2\mu\dot{\varepsilon}_{ij} = \mathscr{D}_{ijkl}\dot{\sigma}_{kl} \tag{6}$$

where, for elastic isotropy,

$$\mathscr{D}_{ijkl} = (\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk})/2-\nu\delta_{ij}\delta_{kl}/(1+\nu)+(\mu/\mu_{eq}^{(p)})(\phi\phi_{ij}\phi_{kl}/\phi_{mn}\sigma_{mn})$$

and then (6) may be inverted to give

$$\dot{\sigma}_{ij}/2\mu = \mathscr{E}_{ijkl}\dot{\varepsilon}_{kl} \tag{7}$$

where, for elastic isotropy,

$$\begin{aligned}\mathscr{E}_{ijkl} &= (\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk})/2+\nu\delta_{ij}\delta_{kl}/(1-2\nu)\\ &\quad-\frac{[\phi_{ij}+\nu\phi_{rr}\delta_{ij}/(1-2\nu)][\phi_{kl}+\nu\phi_{ss}\delta_{kl}/(1-2\nu)]}{\phi_{mn}[\sigma_{mn}\mu_{eq}^{(p)}/\mu\phi+\phi_{mn}+\nu\phi_{tt}\delta_{mn}/(1-2\nu)]}.\end{aligned} \tag{7a}$$

Using the notation

$$\Phi_{ij} = \phi_{ij} + \nu\phi_{rr}\delta_{ij}/(1-2\nu)$$

(7a) is more compactly written

$$\dot{\sigma}_{ij}/2\mu = \dot{\varepsilon}_{ij} + \nu\dot{\varepsilon}_{kk}\delta_{ij}/(1-2\nu) - [\Phi_{ij}\Phi_{kl}\dot{\varepsilon}_{kl}]/[\phi_{mn}(\Phi_{mn} + \sigma_{mn}\mu^{(p)}_{eq}/\phi\mu)]. \quad (8)$$

Combining (1), (2a) and (8), we have the final differential equations

$$\dot{u}_{j,ji}/(1-2\nu) + \dot{u}_{i,jj} - 2[\Phi_{ij}\Phi_{kl}\dot{u}_{k,l}/(\gamma^2 + \phi_{mn}\Phi_{mn})]_{,j} = 0 \quad (9)$$

where

$$\gamma^2 = (\mu^{(p)}_{eq}/\mu)(\phi_{ij}\sigma_{ij}/\phi)$$

is a measure of work-hardening. Note that, for no yielding $1/\gamma^2 = 0$. We refer to (9) as Navier's equations for elasto–plastic flow. Elastic deformations are set as isotropic for convenience; plastic behavior may be anisotropic and compressible.

Special relations

It is easy to show that the plastic strain rates are directly proportional to the total strain rates, viz.

$$\dot{\varepsilon}^{(p)}_{ij} = \phi_{ij}\Phi_{kl}\dot{\varepsilon}_{kl}/(\gamma^2 + \phi_{mn}\Phi_{mn}).$$

In the event of vanishing work-hardening, $\mu^{(p)}_{eq} \to 0$ and we have the dual constitutive relations

$$2\mu\dot{\varepsilon}_{ij} = \dot{\sigma}_{ij} - \nu\dot{\sigma}_{kk}\delta_{ij}/(1+\nu) + \mu\phi\phi_{ij}\dot{\varepsilon}^{(p)}_{eq}/\phi_{mn}\sigma_{mn}$$

$$\dot{\sigma}_{ij}/2\mu = \dot{\varepsilon}_{ij} + \nu\dot{\varepsilon}_{kk}\delta_{ij}/(1-2\nu) - \Phi_{ij}\Phi_{kl}\dot{\varepsilon}_{kl}/\phi_{mn}\Phi_{mn}.$$

Familiar relations obtain, if we set

$$\phi(\sigma_{ij}) = \tau_0 = \sqrt{(s_{ij}s_{ij}/3)};\ s_{ij} = \sigma_{ij} - \sigma_{rr}\delta_{ij}/3$$

then this case of Mises yielding is governed by the relations

$$2\mu\dot{\varepsilon}_{ij} = \dot{\sigma}_{ij} - \nu\dot{\sigma}_{kk}\delta_{ij}/(1+\nu) + (\mu/3\mu^{(p)}_{eq})(s_{ij}s_{kl}\dot{\sigma}_{kl}/3\tau_0^2)$$

$$\dot{\sigma}_{ij}/2\mu = \dot{\varepsilon}_{ij} + \nu\dot{\varepsilon}_{kk}\delta_{ij}/(1-2\nu) - s_{ij}s_{kl}\dot{\varepsilon}_{kl}/[3\tau_0^2(1+3\mu^{(p)}_{eq}/\mu)]$$

or, defining as is usual

$$\dot{\varepsilon}^{(p)}_0 = \sqrt{(\dot{\varepsilon}^{(p)}_{ij}\dot{\varepsilon}^{(p)}_{ij}/3)}$$

we find that

$$\dot{\tau}_0/\dot{\varepsilon}_0 = 6\mu^{(p)}_{eq} \equiv 2\mu^{(p)}_0$$

and hence

$$2\mu\dot{\varepsilon}_{ij} = \dot{\sigma}_{ij} - \nu\dot{\sigma}_{kk}\delta_{ij}/(1+\nu) + (\mu/\mu^{(p)}_0)(s_{ij}s_{kl}\dot{\sigma}_{kl}/3\tau_0^2)$$

$$\dot{\sigma}_{ij}/2\mu = \dot{\varepsilon}_{ij} + \nu\dot{\varepsilon}_{kk}\delta_{ij}/(1-2\nu) - s_{ij}s_{kl}\dot{\varepsilon}_{kl}/[3\tau_0^2(1+\mu^{(p)}_0/\mu)]$$

$$\dot{u}_{j,ji}/(1-2\nu) + \dot{u}_{i,jj} - 2\{s_{ij}s_{kl}\dot{u}_{k,l}/[3\tau_0^2(1+\mu^{(p)}_0/\mu)]\}_{,j} = 0.$$

In the case of perfect plasticity, $\mu_0^{(p)}$ goes to zero as does $s_{ij}\dot{\sigma}_{ij}$ so the first equation becomes

$$2\mu\dot{\varepsilon}_{ij} = \dot{\sigma}_{ij} - \nu\dot{\sigma}_{kk}\delta_{ij}/(1+\nu) + (s_{ij}/\tau_0)(2\mu\dot{\varepsilon}_0^{(p)})$$

and the other two reduce to simpler expressions in an obvious manner.

Characteristic behavior

We are interested in finding the conditions under which (9) may (or may not) be integrated on characteristic surfaces. Such surfaces are denoted by

$$\theta(x) = 0$$

and θ is required to possess all first derivatives $\theta_{,i}$. We seek therefore a statement of conditions on θ; it derives essentially from a geometric argument [7]. The resulting condition is of the form

$$\det[C_{\alpha\beta i}\theta_{,i}] = 0 \tag{10}$$

which is a first-order partial differential equation for $\theta(x)$; it is conceptually soluble within a scale factor, provided that the coefficients in the final expression lead to real values of θ In (10) $C_{\alpha\beta i}$ are coefficients in the original differential equations.

Application to elasto–plastic flow

We turn now to the application of (10) to (9). For purely elastic deformations, (10) becomes

$$[\theta_{,i}\theta_{,i}]^3/(1-2\nu) = 0 \tag{11}$$

or more simply,

$$\text{grad}\,\theta = 0. \tag{11a}$$

The only conclusions we may draw from (11) are that either θ = const., which is trivial, or θ does not exist in real space. In other terms, since there is no characteristic surface of real significance, the equations of elasticity are said to be elliptic. This means that a perturbation of the deformation gradient (or stress) field dies out with distance from the point of disturbance, rather than propagating unabated through the medium; of course, this is merely a rephrasing of the well-known St. Venant principle.

In the more general situation, the condition (11) may be computed after some extensive algebra; the result assumes the form

$$(\theta_{,m}\theta_{,m})[(\gamma^2+\phi_{ij}\phi_{ij})(\theta_{,k}\theta_{,k})^2 - 2\phi_{ik}\phi_{jk}\theta_{,i}\theta_{,j}\theta_{,l}\theta_{,l} + \nu(\phi_{ij}\delta_{ij}\theta_{,k}\theta_{,k} - \phi_{ij}\theta_{,i}\theta_{,j})^2 + (\phi_{ij}\theta_{,i}\theta_{,j})^2] = 0. \tag{12}$$

It is worth noting that the indicial form (12) is deceptive; the equation written *in extenso* is huge. As a result, we have yet to elucidate fully its features. What we expect to observe is substantially the same as what was seen in a series of two-dimensional studies,† namely that the presence of both work-hardening and elastic deformations forces the equation to be elliptic [1]. We further expect plastic compressibility to enhance this effect. Even though this

† It may be noted that (12) easily reduces to the equations developed earlier for the two-dimensional situations studied in [1].

information is not fully in hand, we note that only two sets of orthogonal surfaces are implied by (12) rather than three; this appears kinematically reasonable. Our interest, of course, is not so much to find these surfaces as to elucidate the conditions under which they do not exist.

THREE-DIMENSIONAL ANALYSIS

The procedure we have developed involves use of integral relationships, especially on the boundary of the domain under study. There are certain requirements for using integral equation methods: a reciprocal theorem (e.g. Green's theorem in potential theory, Betti's reciprocal work theorem in elasticity), and a suitable singular solution to the governing differential equations (e.g. Kelvin's problem in elasticity). The combination of these two pieces of information is termed, in classical elasticity, Somigliana's identity; for elasto–plastic flow, we find an extended form of the same relation. By implication, of course, the methods require also that the governing differential equations be elliptic and quasi-linear, properties inherent in the theory of elasto–plastic flow.

Even with this result in hand, there are alternate routes available to the analyst. Our method is analogous to the use of single and double layer potentials for the solution of problems in scalar potential theory [8]. The use of a *boundary constraint equation* contrasts with the work of the Russian mechanicians [9, 10] who follow the classical method. The classical method involves non-physical surface density functions while ours utilizes the physical surface tractions and surface displacements. Hence we term the classical approach indirect and ours, direct. The direct method allows the analyst to obtain stable numerical results for geometries which are not smooth, e.g. having corners and edges. While the classical methods do not seem inherently limited to smooth boundaries [11, 12], they are not numerically stable [13].

In linear elasticity, the advantages to the analyst using the direct method are several. The dimensionality of the problem is reduced by one through the use of the boundary constraint equation, which relates surface tractions to surface displacements. In regions of high gradients, the direct method is appropriate because it provides high resolution of the stresses. Both of these advantages distinguish numerical implementation of the method from finite element and finite difference approaches in three dimensional problems where the entire domain is discretized and the storage requirements are often beyond the capability of an available computer. Finally, the direct method can be automated and treats fully mixed boundary value problems. The method is not restricted to certain geometries but will handle the most general shape including multiply-connected regions. Indeed, the direct method has achieved considerable success in the numerical solution of several linear elastic problems in both two and three dimensions [14–20]. These same features distinguish the direct method from most analytical schemes which are perforce limited to simple geometries and boundary conditions.

Somigliana's identity for the internal displacements

Somigliana's identity for the unknown displacement vector at some point $p(x)$ interior to the body R can be obtained using a form of Betti's reciprocal work theorem and the solution to Kelvin's problem of a point load in an infinite elastic body. If point loads are applied at $p(x)$ in each of three orthogonal directions given by the unit vectors e_i the corresponding displacement at $Q(x)$ is found to be

$$u_i^* = U_{ki}e_k \tag{13}$$

where

$$U_{ki}(p, Q) = [(3-4\nu)\delta_{ik} + r_{,i}r_{,k}]/[16\pi(1-\nu)\mu r]. \tag{14}$$

In (14) the distance between the load point $p(x)$ and the field point $Q(x)$ is given by

$$r = r(p, Q) = \sqrt{[(x_{i|Q} - x_{i|p})(x_{i|Q} - x_{i|p})]}$$

and the derivatives

$$r_{,i} = \partial r/\partial x_{i|Q} = (x_{i|Q} - x_{i|p})/r(p, Q).$$

Utilizing Hooke's law for an elastic, isotropic material the stresses corresponding to the displacements (13) are computed and written in the form

$$\sigma_{ij}^* = \Sigma_{kij}e_k \tag{15}$$

where

$$\Sigma_{kij}(p, Q) = -[(1-2\nu)(\delta_{ki}r_{,j} + \delta_{kj}r_{,i} - \delta_{ij}r_{,k}) + 3r_{,i}r_{,j}r_{,k}]/[8\pi(1-\nu)r^2]. \tag{16}$$

Finally, the traction vector on any surface ∂R surrounding $p(x)$ with normal vector $n_j(Q)$ is given by

$$t_i^* = T_{ki}e_k \tag{17}$$

where

$$T_{ki} = \Sigma_{kij}n_j.$$

Utilizing (3) and (16), the following form of Betti's reciprocal work theorem is easily proven

$$\int_{R-R^*} \sigma_{ij}^* \dot{\varepsilon}_{ij}^{(e)} \, dV = \int_{R-R^*} \varepsilon_{ij}^* \dot{\sigma}_{ij} \, dV. \tag{18}$$

The region R^* is a ball of radius ρ surrounding the load point $p(x)$. This region is deleted due to the singular nature of the Kelvin solution. While reciprocal theorems similar to (18) and its various forms have been given before [21] in terms of total elastic strains and total stresses it is known [22] that no unique relation exists between these tensors. The necessary uniqueness is found only between the strain rates and stress rates in the case of work-hardening material.

Due to the apparent ellipticity of the governing equations—see (9)—the corresponding displacement rates are continuous and possess at least continuous second derivatives. With such conditions the divergence theorem holds and (18) can be written

$$\int_{\partial R + \partial R^*} t_i^* \dot{u}_i \, dS - \int_{R-R^*} \sigma_{ij}^* \dot{\varepsilon}_{ij}^{(p)} \, dV = \int_{\partial R + \partial R^*} u_i^* \dot{t}_i \, dS. \tag{19}$$

The tensor $\dot{\varepsilon}_{ij}^{(p)}$ is related to the velocity gradient via the special relation (shown above):

$$\dot{\varepsilon}_{ij}^{(p)} = \phi_{ij}\Phi_{kl}\dot{u}_{k,l}/[\phi_{mn}(\Phi_{mn} + \sigma_{mn}\mu_{eq}^{(p)}/\mu\phi)].$$

Utilizing the definitions (13), (15), (17) and taking the limit in (19) as $\partial R^* \to 0$ ($\rho \to 0$) the following identity results

$$\dot{u}_k(p)+\int_{\partial R} T_{ki}(p, Q)\dot{u}_i(Q)\,dS_Q = \int_{\partial R} U_{ki}(p, Q)\dot{t}_i(Q)\,dS_Q+\int_R \Sigma_{kij}(p, q)\dot{\varepsilon}_{ij}^{(p)}(q)\,dV_q. \tag{20}$$

In the case $\mu_{eq}^{(p)} \to \infty$, (20) reduces to the usual form in elasticity; we refer to the full equation as an extended form of Somigliana's identity. The plastic term is seen to take the role of a body force in (19) and (20).

The boundary constraint equation

As in previous elastic work (20) is not suitable for numerical solution because the surface displacements and tractions are not both known everywhere on ∂R. The classical—or indirect—method introduces unknown surface densities appropriate to the type of boundary value problem. The direct formulation proceeds to the derivation of the boundary constraint equation by allowing $p(x) \to P(x)$, i.e. the interior point becomes a boundary point. This procedure involves evaluation of the jump in the double-layer potential

$$\psi_k(p) = \int_{\partial R} T_{ki}(p, Q)\dot{u}_i(Q)\,dS_Q \tag{21}$$

for the surface point, $P(x)$. This procedure is well known in the literature; see, e.g. [8–10]. A complication is contained in (20) due to the nature of plasticity, i.e. the tensor $\dot{\varepsilon}_{ij}^{(p)}(q)$ contains not only the current value of the stress field at an interior point $q(x)\varepsilon R$, but also the unknown displacement rate gradient tensor. The development of the boundary constraint equation must therefore be pursued with some care.

Define two vector potentials analogous to single- and double-layer potentials in scalar potential theory [8]. Using the notation in (20) these are, respectively,

$$\phi_k(p) = \int_{\partial R} U_{ki}(p, Q)\mu_i(Q)\,dS_Q \tag{22}$$

$$\psi_k(p) = \int_{\partial R} T_{ki}(p, Q)\lambda_i(Q)\,dS_Q \tag{23}$$

where $\mu_i(Q)$ and $\lambda_i(Q)$ are surface densities. Let $P(x)$ be a surface point which is not at an edge or corner. Then (23) may be written

$$\psi_k(p) = \int_{\partial R} T_{ki}(p, Q)[\lambda_i(Q)-\lambda_i(P)]\,dS_Q+\int_{\partial R} T_{ki}(p, Q)\lambda_i(P)\,dS_Q. \tag{24}$$

If the density $\lambda_i(Q)$ satisfies a Hölder condition on ∂R, it can be shown that the first integral in (24) is continuous as $p(x) \to P(x)$. By detailed investigation it is determined that the second integral has the discontinuity given by

$$\lim_{p\to P}\int_{\partial R} T_{ki}(p, Q)\lambda_i(P)\,dS_Q = -\lambda_k(P)/2+\int_{\partial R} T_{ki}(P, Q)\lambda_i(P)\,dS_Q. \tag{25}$$

The integral in (25) is to be interpreted in the sense of the Cauchy Principal Value. Combining (22) and (24) with the assumption

$$\lim_{p\to P}\psi_k(p) = \psi_k(P)$$

the result is found that

$$\psi_k(P) = -\lambda_k(P)/2 + \int_{\partial R} T_{ki}(P, Q)\lambda_i(Q)\,\mathrm{d}S_Q.$$

By similar procedures the single-layer potential (21) can be shown to be continuous, i.e.

$$\phi_k(P) = \int_{\partial R} U_{ki}(P, Q)\mu_i(Q)\,\mathrm{d}S_Q.$$

Utilizing these results, the identity (25) can be evaluated, and the boundary constraint equation results:

$$\dot{u}_k(P)/2 + \int_{\partial R} T_{ki}(P, Q)\dot{u}_i(Q)\,\mathrm{d}S_Q = \int_{\partial R} U_{ki}(P, Q)\dot{t}_i(Q)\,\mathrm{d}S_Q + \int_R \Sigma_{kij}(P, q)\dot{\varepsilon}_{ij}^{(p)}(q)\,\mathrm{d}V_Q. \quad (26)$$

Note that, to the extent that there is yield, (26) is not strictly a boundary equation; we retain the name, however, as its implications for the theory are significant.

In elasticity, (26) expresses the fact that a relation exists between the surface tractions and the surface displacements. The integral equations which result have been shown [10] to be singular Fredholm equations to which the normal statements of existence and uniqueness may be applied. If the volume integral were a known field, e.g. thermal strains, all of the Fredholm theory would still apply. The nature of plasticity, however, modifies these statements: unknowns appearing in the volume integral eliminate the strict applicability of Fredholm theory. This is not an impediment to implementation of numerical solutions to (26). Even in the elasticity case, when the boundary conditions are of a mixed, or mixed–mixed type the Fredholm Theorems have not been applied. Nevertheless, almost all numerical results to date have involved mixed boundary conditions.

FURTHER REMARKS

We therefore observe the case of combining two established procedures to provide a method for articulating the elasto–plastic behavior of three-dimensional bodies. For the most part, the theoretical basis is already in hand, and there is no evident obstacle to using this approach for solving problems. The most important aspect of the direct method, reduction in problem dimensionality, is substantially retained in the elasto–plastic formulation. Thus, there is considerable operational advantage to the combined method, and work is now under way to implement its development.

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Абстракт—Формулируется общая теория упруго-пластиуеского течения для материалов с упрочнением каканизотропных маки несжимаемых. В виду того, что теория квазилинейная, может быть определена в форме интегральных уравнений. Результат является расширений формой тождества Сомильяно. Когда зти зависимости определенные на границы тела, тогда сокращыется их степенв многомерности. Предыдущий опыт с более простыми материалами указывает, что можно решитв простым способом произвольные задачи.